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COMMENT

Perimeter generating function for row-convex polygons on the rectangular lattice

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Abstract. The perimeter generating function derived recently by Brak *et al* for row-convex polygons on the square lattice is generalized to the rectangular lattice.

A convex polygon on the square lattice is a special case of self-avoiding polygons such that a straight line on the bonds of the dual lattice cuts a convex polygon at most twice. The perimeter generating function for convex polygons on the square lattice was first obtained by Delest and Viennot (1984) and then rederived by Guttmann and Enting (1988), Lin and Chang (1988) and Kim (1988). Lin and Chang (1988) also generalized the result to the rectangular lattice.

Recently Brak *et al* (1990) derived the perimeter generating function for row-convex polygons on the square lattice, which include the convex polygons as a subset. A vertical line on the bonds of the dual lattice cuts a row-convex polygon at most twice. In this comment we generalize their result to the rectangular lattice.

The three-variable generating function for row-convex polygons on the rectangular lattice is defined by

$$G(x, y, z) = \sum_{r,s,k=1}^{\infty} y^{2r}x^{2s}z^{k}P_{r,s,k} = \sum_{r=1}^{\infty} g_r \tag{1}$$

where g_r is the generating function for row-convex polygons whose first vertical row contains r squares, and $P_{r,s,k}$ is the number of row-convex polygons with $2r$ vertical steps, $2s$ horizontal steps and area k. It was shown by Temperley (1956) that g_r satisfies the following equations

$$\begin{aligned}
g_1 &= x^2y^2z + x^2z[g_1+2g_2+3g_3+4g_4+\ldots]\\
g_2 &= x^2y^4z^2 + x^2z^2[2y^2g_1+(1+2y^2)g_2+(2+2y^2)g_3+(3+2y^2)g_4+\ldots]\\
g_3 &= x^2y^6z^3 + x^2z^3[3y^4g_1+(2y^2+2y^4)g_2+(1+2y^2+2y^4)g_3+(2+2y^2+2y^4)g_4+\ldots]\\
g_4 &= x^2y^8z^4 + x^2z^4[4y^6g_1+(3y^4+2y^6)g_2+(2y^2+2y^4+2y^6)g_3\\
&\qquad +(1+2y^2+2y^4+2y^6)g_4+\ldots]
\end{aligned} \tag{2}$$

and similarly for $r > 4$. It follows from (2) that we have the recurrence relation

$$\begin{aligned}
&g_{r+2} - 2z(1+y^2)g_{r+1} + z^2(1+4y^2+y^4)g_r - 2z^3(y^2+y^4)g_{r-1} + z^4y^4g_{r-2}\\
&\qquad = x^2z^{r+2}(1-y^2)^2g_r.
\end{aligned} \tag{3}$$

Temperley (1956) pointed out that these equations are soluble in two special cases of $y=1$ and $z=1$ and he found

$$G(x,1,z)=x^2z(1-z)^3/[1-(x^2+4)z+(x^2+6)z^2-(x^4-x^2+4)z^3+(1-x^2)z^4]. \qquad (4)$$

In the special case of $z=1$, he tried $g_r=A\lambda^r$ and obtained the characteristic equation

$$(\lambda-1)^2(\lambda-y^2)^2-\lambda^2x^2(1-y^2)^2=0. \qquad (5)$$

The perimeter generating function is given by

$$G(x,y,1)=\sum_r g_r=\sum_{j=1}^{4} A_j\lambda_j/(1-\lambda_j) \qquad (6)$$

where $\lambda_1,\ldots,\lambda_4$ are the four roots of (5) and A_j can be determined from the four equations of (2). Temperley did not calculate A_j explicitly. Recently Brak *et al* (1990) considered the special case of $x=y$ (square lattice) and used the computer algebra program MATHEMATICA (Wolfram 1988) to obtain the perimeter generating function $G(y,y,1)$ in a closed form. We shall generalize their result to the rectangular lattice.

The four roots of (5) are

$$\begin{aligned}\lambda_1&=[1-x+(1+x)y^2+(S_+)^{1/2}]/2\\ \lambda_2&=[1-x+(1+x)y^2-(S_+)^{1/2}]/2\\ \lambda_3&=[1+x+(1-x)y^2+(S_-)^{1/2}]/2\\ \lambda_4&=[1+x+(1-x)y^2-(S_-)^{1/2}]/2\end{aligned} \qquad (7)$$

where

$$S_\pm=(1+x^2)(1-y^2)^2\pm 2x(y^4-1).$$

Notice that λ_3, λ_4, S_- can be obtained from λ_1, λ_2, S_+ by the exchange of x with $-x$. In the limit of $y\to 0$, we have

$$\begin{aligned}&\lambda_1\to O(1) \qquad \lambda_2\to O(y^2)\\ &\lambda_3\to O(1) \qquad \lambda_4\to O(y^2)\end{aligned} \qquad (8)$$

while $g_r\to O(y^{2r}x^2)$, therefore we have $A_1=A_3=0$. Following the procedure of Brak *et al* (1990), we define

$$H=\sum_{r=1}^{\infty} rg_r. \qquad (9)$$

Then g_r, H, G are given by

$$\begin{aligned}g_r&=A_2\lambda_2^r+\text{CT}\\ H&=A_2\lambda_2/(1-\lambda_2)^2+\text{CT}\\ G&=A_2\lambda_2/(1-\lambda_2)+\text{CT}\end{aligned} \qquad (10)$$

where CT denotes the conjugate term obtained from the other one by the exchange of x with $-x$. It follows from (2), (3) and (9) that

$$\begin{aligned}g_1&=A_2\lambda_2+A_4\lambda_4=x^2y^2+x^2H\\ g_2-g_1&=A_2\lambda_2(\lambda_2-1)+A_4\lambda_4(\lambda_4-1)=x^2y^2(y^2-1)+x^2(2y^2-1)G.\end{aligned} \qquad (11)$$

We use the computer algebra program REDUCE (Stauffer *et al* 1989) to obtain the following result

$$A_2\lambda_2=[(1-y^2)(1+x)+S_-^{1/2}]A_0 \tag{12}$$

where

$$A_0=[a(ST)^{1/2}+bT^{1/2}+cS^{1/2}+d]/32y^2(1+y^2)\Delta \tag{13}$$

with

$$a=(1-y^2)(2x^2y^2-x^2-15y^2+9)$$

$$b=x^4(1-2y^2)(y^2-1)^2+x^2(17y^6-28y^4+29y^2-10)+3(y^2-1)^2(11y^2+3)$$

$$c=2(y^4-1)(2x^2y^2-x^2-9y^2+9)$$

$$d=2(y^4-1)[2x^4y^4-3x^4y^2+x^4-27x^2y^4+29x^2y^2-10x^2+9(y^2-1)^2]$$

$$\Delta=18(1-y^2)^2-x^2(2-5y^2+2y^4)$$

$$S=1-2(x^2+y^2)+x^4+y^4-12x^2y^2-2x^2y^2(x^2+y^2)+x^4y^4$$

$$T=2(1+x^2)(1-y^2)^2+2(1-y^2)S^{1/2}.$$

The perimeter generating function for the row-convex polygons on the rectangular lattice is given by

$$G(x,y,1)=(1-y^2)[42(1-y^2)^2-2x^2(5-14y^2+5y^4)-6(1-y^2)S^{1/2} -(1-y^2)(17-x^2)T^{1/2}-(ST)^{1/2}]/8\Delta. \tag{14}$$

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